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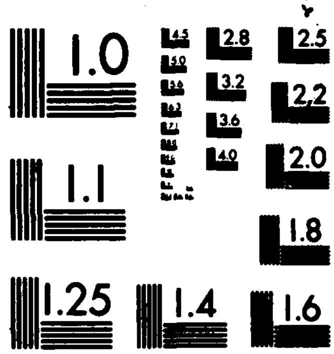
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TECHNICAL REPORT SERIES

AN ALTERNATE APPROACH TO AXIOMATIZATIONS OF

THE VON NEUMANN/MORGENSTERN CHARACTERISTIC FUNCTION

by

Alain A. Lewis and Raghu Sundaram

Technical Report No. 509

March 1987

THE ECONOMICS SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
FOURTH FLOOR, ENCINA HALL
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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A REPORT OF THE
CENTER FOR RESEARCH ON ORGANIZATION EFFICIENCY
STANFORD UNIVERSITY
CONTRACT N00014-K-0216
United States Office of Naval Research

NATIONAL SCIENCE FOUNDATION GRANT DMS-84-10456

THE ECONOMICS SERIES
INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Fourth Floor, Encina Hall
Stanford University
Stanford, California
94305

AN ALTERNATE APPROACH TO AXIOMATIZATIONS OF THE VON NEUMANN/MORGENSTERN CHARACTERISTIC FUNCTION*

by

Alain A. Lewis** and Raghu Sundaram**

1. Introduction

The concept of the characteristic function of a game - that gives us an intuitive idea of the value of a coalition - is of central importance in the theory of N-person cooperative games. In those cases where the players have full knowledge of the structure of a game, in the sense of knowing not only the various parameters but also the payoff functions of the other players, the value of a coalition S, denoted $v(S)$, is defined to be the unique value of the two-person zero-sum game between S and $N - S$. The function thus defined satisfies two properties, and, in general, only these two properties:

P1 $v(\emptyset) = 0$.

P2 If R, $S \in N$, and $R \cap S = \emptyset$, then $v(R \cup S) > v(R) + v(S)$.

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* This work was partially supported by Office of Naval Research Grant N000-14-K-0216 at the Institute for Mathematical Studies in the Social Sciences, Stanford University, Stanford, California and partially by National Science Foundation Grant DMS-84-10456 at Cornell University.

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The first property states that no gain will be forthcoming from non-participation. The second asserts that anything two disjoint coalitions can achieve can be achieved by the union of the two, and possibly even more could be achieved by the latter.

The characteristic function, however, does not tell us anything about the behavior of the players involved. The situation gets even more complicated if we drop the assumption that every player knows every other player's payoff function, and assume merely that he has some, not necessarily correct, perception of these. In this case, we have in addition to the "true" game parametrized by the true payoff functions, the game that different players perceive to exit. In the extreme case, where no player knows anybody's else's payoff function, there are n new games defined, one for each of the players. Associated with each of these games is a characteristic function, v_i, which player i perceives to be the true characteristic function, and upon which his behavior is based. In this situation, the observed behavior can be described in terms of some equilibrium theory, by determining v and all the v_i's.

1.1 The Classical Axiomatization

The idea behind the procedure for determining the subjective characteristic functions is the following: treat the coalitions of players as if they were outcomes and find the preferences of each player among probability mixtures of coalitions. If certain axioms are met, then a utility function exists which, when restricted to pure coalitions is a characteristic function.

Definition 1: A set M is a mixture space if it satisfies the following conditions for all a, b in M and p, q in $[0,1]$:

(i) $a(p)b$ is in M

(ii) $a(p)b = b(1-p)a$

(iii) $a(p)a = a$

(iv) $a(p)b = a(p)c \Rightarrow b = c$

(v) $a(p)[b(q)c] = [a(p/(p+q-pq](p+q-pq)c$.

To elaborate, let A be any set of alternatives. Extend A to the mixture space M of alternatives. Let $\succsim$ represent a binary ordering on M, that satisfies the following axioms:

A1 $\succsim$ is a weak ordering on M.

A2 If $a \succsim b$, then $a(p)c \succsim b(p)c$ for all P in $(0,1)$ and for all c in M, ($a(p)c$ means the alternative a with probability p and the alternative c with probability $(1-p)$).

A3 If $a(p)c \succsim b(p)c$ for some p in $[0,1]$, then $a \succsim b$.

A4 If $a \succ b \succ c$, then there is p in $[0,1]$ such that $a(p)c \sim b$. (where $x \sim y$ iff $x \succsim y$ and $y \succsim x$). Then $\succsim$ gives rise to a family of utility functions, $U(\succsim)$ on M, such that for each u in U the following conditions are met:

U1 $a \succsim b$ iff $u(a) \geq u(b)$.

U2 $u(a(p)b) = p \cdot u(a) + (1 - p) \cdot u(b)$.

U3 u' is in U iff u' is a positive affine transformation of u.

Assume now that A is the set of permissible coalitions of a finite N-person cooperative game and M is, therefore, a probability mixture of coalitions. Assume further that the payoff to each player in a coalition is the average payoff to the members of that coalition and that if a player chooses a probability mixture $a(p)b$, then he gets a payoff from a according to the same rule with probability p and from b with probability $(1 - p)$. Finally, we impose the additional axiom.

A5 If a,b are in A, and $a \cap b = \emptyset$, then $a \cup b \succsim a(p)b$ for $p = |a|/|a \cup b|$.

Pick any u from the family of utility funtions satisfying A1-A5. Define $C(u)$ to be the class of set functions: $v\colon M \to \mathbb{R}$,

$$v(R) = c|R| \cdot [u(R) - u(\emptyset)] + \sum_{i \in R} d_i, \; c > 0, \; d_i \text{ in } \mathbb{R}, \text{ all } i.$$

It turns out that v is a characteristic function that v' is in $C(u)$ iff v' is a positive linear transform of v. More importantly Luce and Adams [1956] have shown the following:

<u>Theorem 1.1</u>: Let v be a given characteristic function. Extend v to the mixture space of coalitions by the definition $v(a(p)b) = p \cdot v(a) + (1 - p) \cdot v(b)$. Define $\succsim$ by the rule $a \succsim b$ iff $[v(a)/|a|] > [v(b)/|b|]$. Then $\succsim$ satisfies axioms A1-A5, and $v \in C(u)$ for $u \in U(\succsim)$.

In effect, then, a preference relation over the mixture space of coalitions that satisfies a given set of axioms generates an S-equivalent class of characteristic function. Moreover, if a preference relation over the mixture space of coalitions is based rationally on a subjective characteristic function v, this preference relation will meet the axioms and generate a class of characteristic functions S-equivalent to v.

In this paper, we examine the relationship between axiomatic structure and the characteristic function. In particular we shall attempt to build up the characteristic function axiomatically, but using weaker axioms than those used above. One axiom we shall be doing away with altogether is that of defining the mixture space to extend to all real probabilities - rational and irrational. Irrational probabilities hold little intuitive appeal - it is hard to appreciate the value of such a probability and difficult to visualize making sensible choices between alternatives when such numbers are involved. Moreover, the restriction to rational probabilities carries an added computational feature: Every rational number is a recursive or computable real number and thus can be approximated recursively by a number theoretic function whose values can be computed by a device of artificial intelligence known as a Turing Machine. Additionally, the rationals as a set of recursive real numbers is recursively enumerable; what this means it that there is an effective procedure to list the entire set of rationals by an algorithm that codes the simulation of a Turing Machine. A reference for recursive real numbers is the article by Rice [1954] and an introduction to computability and Turing Machines can be found in Rogers [1967]. Based on Shepherdson's [1980] paper we shall examine various axiomatic constructions of the characteristic function using only rational probabilities, while simulataneously weakening the other axioms.

2. The Framework

Let K be a multiplier set consisting of all rationals in $[0,1]$. Let A be the set of permissible coalitions. Extend A to X, the K-mixture space of coalitions.

Definition 2: A multiplier set K is a subset of $[0,1]$ such that 0 and $1/2$ belong to K, and for all x,y in K, $(1 - x)$ and xy are in M.

Definition 3: If A is an arbitrary set of alternatives then X is the K-mixture set generated by A if K is a multiplier set and if the conditions (i)-(v) in the definition of a mixture set are satisfied for all a,b in A and all p,q in K.

Let $\succsim$ be a binary relation on X. For ease of future reference, we shall lay down a grand list of axioms, subsets of which we shall employ subsequently.

X1 X is completely ordered by $\succsim$

X2 For all R,S,T in X, the sets

$G(T) = \{m$ in K: $R(m)S \succsim T\}$, and

$L(T) = \{m$ in K: $R(m)S \precsim T\}$

are closed in K.

X3 For all R,R' in X, if $R \sim R'$, then $R(1/2)S \sim R'(1/2)S$.

X3' For all R,R' in X, if $R \succsim R'$, then $R(1/2)S \succsim R'(1/2)S$.

X4 For all R,S in X, and for all m,n in K, $R \succsim S$, $m > n$ implies $R(m)S \succsim R(n)S$

XvN For all R,S,T in X, if $R \succ S \succ T$, then there exist m,n in K, $m < 1$, $n > 0$, such that $R(m)T \succ S \succ R(n)T$.

X5 If R,S in X, and $R \cap S = \emptyset$, then $R \quad S \succsim R(m)S$ for $m = [|R| \; / \; |R \cup S|]$.

Definition 4: A measurable utility function is a function u: $X \to \mathbb{R}$, that is order preserving ($R \succsim S$ iff $u(R) > U(S)$) and linear ($U(R(m)S) = m \cdot u(R) + (1 - m) \cdot u(S)$), for any $m \in K$.

Theorem 2.1: A necessary and sufficient condition for the existence of a measurable utility function on X is that axioms X1, X2 and X3' be satisfied. Moreover, this utility function is unique up to a positive affine transformation.

Proof: See Shepherdson [1980] Theorem 2.3. $\square$

Suppose, now, that in addition to these axioms we impose Axiom X5. If u is any member of the family of utility functions on X whose existence Theorem 2.1 asserts, it is immediate that

$$u(R \cup S) > [|R|/|R \cup S|]\, u(R) + [|S|/|R \cup S|]\, u(S).$$

Define $C(u)$ to be the class of set functions $v\colon X \to \mathbb{R}$ such that

$$v(R) = c \cdot |R| \cdot [u(R) - u(\emptyset)] + \sum_{i \in R} d_i, \quad c > \emptyset,\ d_i \text{ in } \mathbb{R}, \text{ all } i.$$

We can easily establish

Theorem 2.2: (a) If v is in $C(u)$, then v is a characteristic function.

(b) v' is in $C(u)$ if, and only if, v' is a positive affine transformation of v.

Proof: Clearly $v(\emptyset) = 0$.

Suppose $R, S \in X$ and $R \cap S = \emptyset$.

$$V(R \cup S) = c \cdot |R \cup S| \cdot [u(R \cup S) - u(\phi)] + \sum_{i \in R \cup S} d_i$$

$$> c \cdot |R \cup S| \, [(|R|/|R \cup S|)\, u(R) +$$

$$(|S|/|R \cup S|)\, u(S)] + \sum_{i \in R \cup S} d_i$$

$$= v(R) + v(S).$$

This establishes (a). Suppose v' is in $C(u)$. Then,

$$v'(R) = c'|R| \, [u(R) - u(\phi)] + \sum_{i \in R} d_i'$$

$$v(R) = c|R| \, [u(R) - u(\phi)] + \sum_{i \in R} d_i$$

so clearly,

$$v'(R) = a \cdot v(R) + \sum_{i \in R} b_i$$

where $a = c/c'$ and $b_i = (cd'_i - c'd_i)/c$.

Conversely, suppose $v'(R) = a'v(R) + \sum_{i \in R} b'_i$.

Then, $v'(R) = c'|R| \, [u(R) - u(\phi)] + \sum_{i \in R} d'_i$

where $c' = a'c > \emptyset$, and $d'_i = (a'd_i + b'_i)$,

so v' is in $C(u)$. $\square$

Theorem 2.2 shows that it is possible to build up a characteristic function from weaker assumptions on the preference ordering on the mixture space. What is more interesting is that we can prove an exact analog of Theorem 1.1 for X and thus establish the entire axiomatization on new grounds:

<u>Theorem 2.3</u>: Let v be a given characteristic function. Extend v to the M-mixture space of coalitions X by defining $v(R(m)S) = m\ v(R) + (1 - m)\ v(S)$. Define the relation $\succsim$ on X by $R(m)S \succsim R'(m')S'$ if, and only if, $m(v(R)/|R|) + (1 - m)\ (v(S)/|S|) \geq m'\ (v(R')/|R;|) + (1 - m')\ (v(S')/|S'|)$. Then $\succsim$ satisfies X1, X2, X3' and X5. Further, if U denotes the class of utility functions generated by $\succsim$ defined as above, $v \in C(u)$ for $u \in U$.

<u>Proof:</u> Completeness and transitivity are obvious, since the relation $\geq$ on $\mathbb{R}$ satisfies both. Define

$$G(T) = \{m \text{ in } M\text{: } R(m)S \succsim T\}, \text{ and}$$

$$L(T) = \{m \text{ in } M\text{: } R(m)S \precsim T\}.$$

Let $\{m_k\}_{k=1}^{\infty}$ be a sequence in $G(T)$, with a limit m in M. Then, for all k,

$$m_k(v(R)/|R|) + (1 - m_k)\ (v(S)/|S|) \geq v(T)/|T|.$$

In the limit, the same inequality holds with m replacing m_k, so $m \in G(T)$. Hence $G(T)$ is closed. Similarly $L(T)$ is closed. Next, suppose $R \succsim S$. Then, $(v(R)/|R| \geq (v(S)/|S|)$. Multiplying both sides by m and adding $(1 - m)\ (v(T)/|T|)$ does not change the inequality, so indeed $R(m)T \succsim S(m)T$. Fourth, suppose $R \cap S = \emptyset$. Since v is a charateristic function, $v(R) + v(S) \leq v(R \cup S)$. So,

$(|R|/|R \cup S|)(v(R)/|R \cup S|) + (|S|/|R \cup S|)(v(S)/|S|) < (v(R \cup S)/|R \cup S|)$, which means $R(|R|/|R \cup S|)S \succsim R \cup S$. Hence, all the axioms satisfied and there exists a family U of utility functions on X, such that if u and u' are both in U, then u' is a positive affine transformation of u. It is easy to see that, if we define $C(u)$ as before for u in U, then $C(u)$ is invariant to the choice of the particular u. Now, note that if we define $u(R(m)S) = m(v(R)/|R|) + (1 - m)\cdot(v(S)/|S|)$, then this is a measurable utility function on X, and is hence in U. Defining $C(u)$ on this basis, then $w \in C(u) \rightarrow w(R) = c\cdot|R|[v(R)/|R| - 0] + \sum_{i \in R} d_i$, $c > 0$, $d_i \in \mathbb{R}$ which says just that v is S-equivalent to w, and is therefore in $C(u)$.

$\square$

Theorem 2.3 completes the alternate axiomatization in the development of the characteristic function. That this is a non-trivial generalization is shown by Lemma 2.4, and the (obvious) fact that the converse of the Lemma is false, since the new axioms require the preference ordering only over rational mixtures.

<u>Lemma 2.4</u>: Axioms A1-A4 imply axioms X1, X2 and X3'.

<u>Proof</u>: Axioms A1 and X1 are the same except that the latter requires a weak ordering over only rational mixtures, so A1 → X1. Axiom X3' is but a special case of axiom A2 for $p = 1/2$. All that needs to be shown, then, is that axioms A1-A4 imply that the sets $G(T) = \{m \text{ in } M\colon R(m)S \succsim T\}$, and $L(T) = \{m \text{ in } M\colon R(m)S \precsim T\}$ are

closed in M. A little thought shows that in the above context, all the possible cases can be covered in the following four cases.

(i) $R \succ T$, $S \succsim T$ (iii) $R \sim S \sim T$

(ii) $R \prec T$, $S \precsim T$ (iv) $R \succ T \succ S$.

In case (i), we have

$$R \succ T \Rightarrow R(m)S \succ T(m)S \quad \text{for all } m \text{ in } [0,1]$$

$$= S(1 - m)T$$

$$\succsim T(1 - m)T = T,$$

and for $m = 0$, clearly $R(m)S \sim T(m)S$. So, $G(T) = M$ and $L(T) = \emptyset$, which are trivially closed in M.

Cases (ii) and (iii) are similarly disposed of. Case (iv) is a bit more involved: From A4, thre is m in $(0,1)$ such that $R(m)S \sim T$.

$$\text{So} \quad [R(m)S](m')S \sim T(m')S \quad \text{for all } m' \text{ in } [0,1]$$

$$\precsim T$$

or $R(mm')S \precsim T$ for all m' in $[0,1]$. Again

$$[R(m)S](m'')R \sim T(m'')R \quad \text{all } m'' \text{ in } [0,1]$$

$$\succsim T$$

or $R(mm'' + (1 - m''))S \sim T$ for all m'' in $[0,1]$. Since $mm' < m$ for all m' in $[0,1]$, and $mm'' + (1 - m'') > m$ for all m'' in $[0,1]$ we see that $G(T) - \{n \text{ in } M\colon n > m\}$, and $L(T) = \{n \text{ in } M\colon n < m\}$.

Regardless of whether m is in M or not, these sets are clearly closed in M. $\square$

Another axiomatic system on which we could base the characteristic function is provided by the following theorem and similar constructions of those of Theorem 2.4. The details are left to the reader.

Theorem 2.5: It is necessary and sufficient for the existence of a measurable utility function on X that the following axioms be satisfied: X1, X3', X4 and XvN.

Proof: By Shepherdson [1980], Theorem 5.2 axioms X4 and XvN together imply X2, hence the necessity and sufficiency. $\square$

There are then, at least two possible axiomatizations that are weaker - and, therefore more general - than the classical axiomatization. Note also that if we wish to build up the characteristic function as (essentially) an affine transform of a measurable utility function when restricted to pure coalitions, the above conditions are, in some sense, "minimal". The conditions for the existence are both necessary and sufficient. In the following section we will examine possible methods of further weakening the axioms on the preference ordering. In particular, we shall look at (a) the necessity of completeness of the ordering and (b) the necessity of measurability as a property of the utility function. These topics are intimately linked - completeness is necessary for the utility function to be measurable.

A final remark before we close this section Theorems 1.1 and 2.3 showed that if we started with a characteristic function v, and defined an ordering on X using v in a rational manner, the ordering would satisfy all the axioms required for the existence of a utility function. This utility function, would in turn generate a family of characteristic functions S-equivalent to v. These theorems require no more than the existence of the characteristic function.

3. Further Generalizations

The alternate set of axioms we have presented reduce tremendously the requirements each player's preference ordering must satisfy. At one level, however, this does not seem to be much of an improvement: employing a mixture set that is denumerable rather than uncountable does not take away the fact that we are still insisting on a very large number of comparisons. Two ways to improve this situation suggest themselves. First, we could try developing the utility function on the basis of preference axioms that do not impose completeness and then develop the characteristic function on this basis, possibly in a manner similar to the method we adopted above. Alternately, we could abandon the axiomatic definition of the utility function, and attempt to develop the characteristic function with the utility function as the primal concept, and examine the alternate utility functions we could work with. This approach seems less satisfying intellectually, but carries the advantage of helping us understand the precise nature of the link between the characteristic function of a game, and the attitude towards risk of the players - which is, in fact, one of the most important things the utility function tells us. We examine both these approaches.

3.1 Incompleteness of Preferences and The Characteristic Function

Completeness, as was stated in the previous section, is necessary for the existence of a utility function on the mixture space that is order preserving and linear. Without it, as we shall note in Theorem 3.1, we lose the order preserving property. This in itself does not seem very important. It seems plausbile that a one-way utility function (defined as a function $U: X \to \mathbb{R}$ that satisfies the condition $u(R) > u(S)$ if $R \succ S$) might serve our purpose if we insist on completeness of preferences at least on the space of pure coalitions. As one might expect, completeness on this subspace turns out to be necessary for the definition of a characteristic function via a utility function - an interesting link between the characteristic function of the game and the preferences of its players. We shall establish two results before continuing this discussion:

Theorem 3.1: Let X be the set of simple rational-valued probability measures on a finite nonempty set A. Let $\succ$ be a binary relation on X that satisfies, for all $P, R, S, T,$ in X:

P1 $\succ$ is irreflexive.

P2 If p is a rational in $(0,1)$, then $P \succ R$ and $S \succ T$ implies $P(p)S \succ R(p)T$.

<u>P3</u> $P \succ R$, $S \succ T$ implies there exists a rational p in $(0,1)$ such that $P(p)T \succ R(p)S$.

Then there exists a one-way utility on X.

<u>Proof</u>: See Shepherdson, [1980], Theorem 6.1. $\square$

The interpretation of these sets is the same as before, but with a slight twist: while A remains the space of pure coalitions, X is the set of all functions $P: A \to Q$ satisfying $P(A(i)) > 0$, for all i, and $\sum_{i \in I} P(A(i)) = 1$. (Here, $A(i)$ refers to the i'th member of A and where I is an index of the members of A.)

<u>Lemma 3.2</u>: In order for a characteristic function to be defined via a utility function, it is necessary that A be completely ordered.

<u>Proof</u>: The counter-example is constructed as follows: Let $N = \{1,2,3\}$ and $A = \{1,2,3,12,13,23,123\}$. Define an incomplete ordering on A by: $1 \succ 2 \succ 3$, $123 \succ 12 \succ 13 \succ 23$. For R,S in X such that R,S do not assign probability 1 to any element of A, Let $R \overset{=}{\underset{\sim}{\succ}} S$. Also, for R in $\{X - A\}$, let $R \succ 1$, and $R \succ 123$. Note that axioms P1-P3 are satisfied and there is a one way utility on X. In fact there are several and it is not too difficult to come out with one that will serve our purpose. Consider, for example, the function given by:

$u(1) = 4$, $u(2) = 3$, $u(3) = 2$,

$u(123) = 4$, $u(12) = 3$, $u(13) = 2$, $u(23) = 1$,

$u(R) - 5$ for all R not in A, $u(\emptyset) = \emptyset$.

(We assume that every set is preferred to the empty set.) Clearly, for any positive affine transform of the restriction of u to A, $v(13) < v(1) + v(3)$. Thus the restriction cannot be a characteristic function.

$\square$

Theorem 3.1 and Lemma 3.2 require conditions that are far weaker than any we imposed in the previous section. We have paid for this weakening in the loss of the most powerful features of the utility function: its order preserving and linearity properties. The first is not very important if the conditions of Lemma 3.2 are satisfied. The second loss is crucial.

Recall that a characteristic function must satisfy the condition of super-additivity. This was ensured under the previous axiomatization by the combination of Axiom X5 and the linearity of the utility function. Without linearity, the mere imposition of Axiom X5 is insufficient to do the trick. To see this rigourously, let $f: \mathbb{R} \to \mathbb{R}$ be a transform of u. Then, if F is monotone, we know from Theorem 3.1 and the additional imposition of Axiom X5 that

$$R \succ S \Rightarrow f(u(R)) > f(u(S)), \qquad \text{and}$$

$$\text{if } p = |R|/|R \cup S|), \ f(u(R \cup S)) > f(u(R(p)S))$$

For the transform f to be a characteristic function, we need

$$f(u(\emptyset)) = 0, \qquad \text{and}$$

$$f(u(R \cup S)) > f(u(R)) + f(u(S)).$$

Note that previously we had defined f to be a linear transform of $|R| \cdot u(R)$, and linearity of u guaranteed that this was a characteristic function. It is obvious that, in this case, there is no unique functional f that will, for all one-way utility functions satisfying the Axioms of Theorem 3.1, ensure that the transformation is a characteristic function.

What is required, therefore, is an axiomatization that will yield linearity of the utility function through preferences that are only partially complete. At first sight, this seems an impossible requirement, for as we have noted before, a measurable utility function cannot exist that is not based on completeness. Completeness, however, provides two properties - order-preservation and linearity - only the second of which we need. It would be sufficient for our purposes if the utility function were one-way and linear. But we know of no procedure to construct such a utility within the framework of m-mixture extensions of finite sets of coalitions. Therefore an extension of our axiomatization to incomplete orders on A remains an open problem at this time.*

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* There is of course, the result of B. Peleg that actually constructs a one-way utility from a partially ordered topological space of alternatives where the partial ordering on the space is assumed to be continuous, separable, spacious and strict (cf. B. Peleg [1970] for the

3.2 Attitudes to Risk and The Characteristic Function

In Classical utility theory an agent's attitude to risk is reflected in the curvature of his/her utility function. To illustrate, consider the following problem facing the agent: The agent has the choice of entering or not entering a lottery. If the gambling option is chosen, the agent receives a payoff of x with probability p, and y with probability $(1 - p)$. If the agent chooses not to gamble, a certain payoff of $(px + (1 - p)y)$ is received.

The agent will choose to gamble, be indifferent to the lottery, or gamble according as the utility received from gambling is greater, equal to, or less than the utility received from the certain payoff. If x and y are reasonably close together, then it is likely that the shape of the agent's utility function will not alter dramatically (in the sense of changing shape from concave to convex). Under the assumption that the agent possesses a measurable utility function, the agent will gamble if

$$u(px + (1 - p)y) < p \cdot u(x) + (1 - p) \cdot u(y).$$

The agent will not gamble if the inequality is reversed, and will be indifferent to the two options if equality holds. In obviously suggestive terminology, we that the agent is risk loving in the first case, in the second that the agent is risk-averse, and in the last that

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definitions of these terms). But the function provided by Peleg's construction is not linear and we cannot find the needed additional requirements to insure linearity in Peleg's constructions at this time. Peleg's construction does not work for weakly (i.e., non-strict) partially ordered topological spaces. (cf. Remark 4.2 of Peleg [1970], p. 95) he is risk-neutral. It is clear that risk-aversion implies local concavity of the agent's utility function, risk-loving implies local convexity, and risk-neutrality implies that the utility function is locally affine.

Since we have (implicitly) seen much of the relation between the utility and characteristic functions, we will not dwell on the obvious here. What we will mention is a rather surprising connection between the two concepts. Let us assume the measurability of the utility functions of the players in a game and proceed to define the characteristic function from the utility function exactly as in Section 2. Then it is the case that the agents cannot be strictly risk-loving uniformly on their domains. A simple computation shows that the violation of this condition will result in the falsification of Axiom X5, and therefore the transform of the utility function no longer satisfies the superadditivity condition. Hence, such a transform cannot be a characteristic function.

4. Conclusions

We have seen that the classical model has a few shortcomings that make the axiomatization somewhat unrealistic. The alternate axiomatization that we presented, removed the requirement that the players' preferences had to be defined over all probability mixtures, even when these probabilities were irrational, replacing it with a system where we only required the preferences to be complete over a mixture space of rationals. It was shown that all the results of the classical framework continued to hold under this weaker framwork. Avenues of further generalization were explored with not very heartening results. It does not seem possible to build up the axiomatic system on preferences which are complete only on a very limited subset of the mixture space - the set of pure coalitions. An interesting link between the characteristic function and the attitude of risk to the players was observed where it was shown that it is not possible that the characteristic function could be defined from the preferences of players who are strictly risk loving uniformly on the domain of their utility functions.

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